Computer Vision - Lecture 7

Segmentation as Energy Minimization

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Course Outline

• Image Processing Basics

• Segmentation
  ➢ Segmentation and Grouping
  ➢ Segmentation as Energy Minimization

• Recognition
  ➢ Global Representations
  ➢ Subspace representations

• Local Features & Matching

• Object Categorization

• 3D Reconstruction

• Motion and Tracking
Recap: Image Segmentation

• Goal: identify groups of pixels that go together
Recap: K-Means Clustering

- Basic idea: randomly initialize the $k$ cluster centers, and iterate between the two steps we just saw.
  1. Randomly initialize the cluster centers, $c_1, \ldots, c_k$
  2. Given cluster centers, determine points in each cluster
     - For each point $p$, find the closest $c_i$. Put $p$ into cluster $i$
  3. Given points in each cluster, solve for $c_i$
     - Set $c_i$ to be the mean of points in cluster $i$
  4. If $c_i$ have changed, repeat Step 2

- Properties
  - Will always converge to some solution
  - Can be a “local minimum”
    - Does not always find the global minimum of objective function:
      $$\sum_{\text{clusters } i} \sum_{\text{points } p \text{ in cluster } i} \|p - c_i\|^2$$

Slide credit: Steve Seitz
Recap: Expectation Maximization (EM)

- **Goal**
  - Find blob parameters $\theta$ that maximize the likelihood function:
    \[
    p(data|\theta) = \prod_{n=1}^{N} p(x_n|\theta)
    \]

- **Approach:**
  1. **E-step:** given current guess of blobs, compute ownership of each point
  2. **M-step:** given ownership probabilities, update blobs to maximize likelihood function
  3. Repeat until convergence

Slide credit: Steve Seitz
Recap: EM Algorithm

- **Expectation-Maximization (EM) Algorithm**
  - **E-Step:** softly assign samples to mixture components
    \[
    \gamma_j(x_n) \leftarrow \frac{\pi_j \mathcal{N}(x_n | \mu_j, \Sigma_j)}{\sum_{k=1}^{K} \pi_k \mathcal{N}(x_n | \mu_k, \Sigma_k)}
    \quad \forall j = 1, \ldots, K, \quad n = 1, \ldots, N
    \]
  - **M-Step:** re-estimate the parameters (separately for each mixture component) based on the soft assignments
    \[
    \hat{N}_j \leftarrow \sum_{n=1}^{N} \gamma_j(x_n) = \text{soft number of samples labeled } j
    \]
    \[
    \hat{\pi}_j^{\text{new}} \leftarrow \frac{\hat{N}_j}{N}
    \]
    \[
    \hat{\mu}_j^{\text{new}} \leftarrow \frac{1}{\hat{N}_j} \sum_{n=1}^{N} \gamma_j(x_n) x_n
    \]
    \[
    \hat{\Sigma}_j^{\text{new}} \leftarrow \frac{1}{\hat{N}_j} \sum_{n=1}^{N} \gamma_j(x_n) (x_n - \hat{\mu}_j^{\text{new}})(x_n - \hat{\mu}_j^{\text{new}})^T
    \]

Slide adapted from Bernt Schiele
MoG Color Models for Image Segmentation

- **User assisted image segmentation**
  - User marks two regions for foreground and background.
  - Learn a MoG model for the color values in each region.
  - Use those models to classify all other pixels.
  ⇒ Simple segmentation procedure
    (building block for more complex applications)
Recap: Mean-Shift Algorithm

- **Iterative Mode Search**
  1. Initialize random seed, and window \( W \)
  2. Calculate center of gravity (the “mean”) of \( W \):\[
  \sum_{x \in W} x H(x)
  \]
  3. Shift the search window to the mean
  4. Repeat Step 2 until convergence
Recap: Mean-Shift Clustering

- Cluster: all data points in the attraction basin of a mode
- Attraction basin: the region for which all trajectories lead to the same mode
Recap: Mean-Shift Segmentation

- Find features (color, gradients, texture, etc)
- Initialize windows at individual pixel locations
- Perform mean shift for each window until convergence
- Merge windows that end up near the same “peak” or mode

Slide credit: Svetlana Lazebnik
Back to the Image Segmentation Problem...

- **Goal:** identify groups of pixels that go together

- **Up to now, we have focused on ways to group pixels into image segments based on their appearance...**
  - Segmentation as clustering.

- **We also want to enforce region constraints.**
  - Spatial consistency
  - Smooth borders
Topics of This Lecture

- **Segmentation as Energy Minimization**
  - Markov Random Fields
  - Energy formulation

- **Graph cuts for image segmentation**
  - Basic idea
  - s-t MinCut algorithm
  - Extension to non-binary case

- **Applications**
  - Interactive segmentation
Markov Random Fields

- Allow rich probabilistic models for images
- But built in a local, modular way
  - Learn local effects, get global effects out

Observed evidence

Hidden “true states”

Neighborhood relations

Slide credit: William Freeman
MRF Nodes as Pixels

Original image  Degraded image  Reconstruction from MRF modeling pixel neighborhood statistics

\[ \Phi(x_i, y_i) \]

\[ \Psi(x_i, x_j) \]

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Network Joint Probability

\[ P(x, y) = \prod_{i} \Phi(x_i, y_i) \prod_{i,j} \Psi(x_i, x_j) \]

- Scene
- Image
- Image-scene compatibility function
- Scene-scene compatibility function
- Local observations
- Neighboring scene nodes

Slide credit: William Freeman
Energy Formulation

- Joint probability
  \[ P(x, y) = \prod_i \Phi(x_i, y_i) \prod_{i,j} \Psi(x_i, x_j) \]

- Maximizing the joint probability is the same as minimizing the negative log
  \[ -\log P(x, y) = -\sum_i \log \Phi(x_i, y_i) - \sum_{i,j} \log \Psi(x_i, x_j) \]
  \[ E(x, y) = \sum_i \phi(x_i, y_i) + \sum_{i,j} \psi(x_i, x_j) \]

- This is similar to free-energy problems in statistical mechanics (spin glass theory). We therefore draw the analogy and call \( E \) an energy function.

- \( \phi \) and \( \psi \) are called potentials.
Energy Formulation

- **Energy function**
  \[
  E(x, y) = \sum_i \phi(x_i, y_i) + \sum_{i,j} \psi(x_i, x_j)
  \]
  Single-node potentials  
  Pairwise potentials

- **Single-node potentials** \( \phi \)
  - Encode local information about the given pixel/patch
  - How likely is a pixel/patch to belong to a certain class (e.g. foreground/background)?

- **Pairwise potentials** \( \psi \)
  - Encode neighborhood information
  - How different is a pixel/patch’s label from that of its neighbor? (e.g. based on intensity/color/texture difference, edges)
Energy Minimization

- **Goal:**
  - Infer the optimal labeling of the MRF.

- **Many inference algorithms are available, e.g.**
  - Gibbs sampling, simulated annealing
  - Iterated conditional modes (ICM)
  - Variational methods
  - Belief propagation
  - Graph cuts

- **Recently, Graph Cuts have become a popular tool**
  - Only suitable for a certain class of energy functions
  - But the solution can be obtained very fast for typical vision problems (~1MPixel/sec).

see lecture Machine Learning!
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• Segmentation as Energy Minimization
  ➢ Markov Random Fields
  ➢ Energy formulation

• Graph cuts for image segmentation
  ➢ Basic idea
  ➢ s-t Mincut algorithm
  ➢ Extension to non-binary case

• Applications
  ➢ Interactive segmentation
Graph Cuts for Optimal Boundary Detection

- Idea: convert MRF into source-sink graph

Minimum cost cut can be computed in polynomial time (max-flow/min-cut algorithms)

$$w_{pq} = \exp \left\{ -\frac{\Delta I_{pq}}{2\sigma^2} \right\}$$

[Boykov & Jolly, ICCV'01]

Slide credit: Yuri Boykov
Simple Example of Energy

\[ E(L) = \sum_p D_p(L_p) + \sum_{pq \in N} w_{pq} \cdot \delta(L_p \neq L_q) \]

- **Regional term**
  \[ D_p(L_p) \]

- **Boundary term**
  \[ w_{pq} = \exp \left\{ -\frac{\Delta I_{pq}}{2\sigma^2} \right\} \]

**t-links**

**n-links**

A cut

\[ L_p \in \{s, t\} \]

(binary object segmentation)

Slide credit: Yuri Boykov

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Adding Regional Properties

Regional bias example

Suppose $I^s$ and $I^t$ are given “expected” intensities of object and background

$$D_p(s) \propto \exp \left( - \| I_p - I^s \|^2 / 2\sigma^2 \right)$$

$$D_p(t) \propto \exp \left( - \| I_p - I^t \|^2 / 2\sigma^2 \right)$$

NOTE: hard constrains are not required, in general.

[Boykov & Jolly, ICCV ’01]
Adding Regional Properties

"expected" intensities of object and background $I^s$ and $I^t$ can be re-estimated

$D_p(t) \propto \exp \left( -\| I_p - I^s \|^2 / 2\sigma^2 \right)$

$D_p(s) \propto \exp \left( -\| I_p - I^t \|^2 / 2\sigma^2 \right)$

EM-style optimization

Slide credit: Yuri Boykov
Adding Regional Properties

- More generally, regional bias can be based on any intensity models of object and background.

\[ D_p(L_p) = -\log p(I_p|L_p) \]

Given object and background intensity histograms.

Slide credit: Yuri Boykov

[Boykov & Jolly, ICCV’01]
How to Set the Potentials? Some Examples

• Color potentials
  - e.g., modeled with a Mixture of Gaussians
    \[ \pi(x_i, y_i; \theta_\pi) = \log \sum_k \theta_\pi(x_i, k) P(k \mid x_i) N(y_i; \bar{y}_k, \Sigma_k) \]

• Edge potentials
  - E.g., a “contrast sensitive Potts model”
    \[ \phi(x_i, x_j, g_{ij}(y); \theta_\phi) = -\theta_\phi^T g_{ij}(y) \delta(x_i \neq x_j) \]
    where
    \[ g_{ij}(y) = e^{-\beta \|y_i - y_j\|^2} \quad \beta = 2 \cdot \text{avg} \left( \|y_i - y_j\|^2 \right) \]

• Parameters \( \theta_\pi, \theta_\phi \) need to be learned, too!
Example: MRF for Image Segmentation

- **MRF structure**

  ![Graphical representation of MRF structure]

  - **Unary potentials**
  - **Pairwise potentials**

- **Data (D)**
- **Unary likelihood**
- **Pair-wise Terms**
- **MAP Solution**
Topics of This Lecture

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  - Markov Random Fields
  - Energy formulation

- **Graph cuts for image segmentation**
  - Basic idea
  - s-t Mincut algorithm
  - Extension to non-binary case

- **Applications**
  - Interactive segmentation
How Does it Work? The s-t-Mincut Problem

Graph \((V, E, C)\)
- Vertices \(V = \{v_1, v_2, ..., v_n\}\)
- Edges \(E = \{(v_1, v_2), ..., \}\)
- Costs \(C = \{c_{(1,2)}, ..., \}\)
The s-t-Mincut Problem

What is an st-cut?

An st-cut \((S,T)\) divides the nodes between source and sink.

What is the cost of a st-cut?

Sum of cost of all edges going from S to T

\[5 + 2 + 9 = 16\]

Slide credit: Pushmeet Kohli
The s-t-Mincut Problem

What is an st-cut?

An st-cut \((S, T)\) divides the nodes between source and sink.

What is the cost of an st-cut?

Sum of cost of all edges going from \(S\) to \(T\)

What is the st-mincut?

st-cut with the minimum cost

2 + 1 + 4 = 7

Slide credit: Pushmeet Kohli
How to Compute the s-t-Mincut?

Solve the dual maximum flow problem

Compute the maximum flow between Source and Sink

Constraints
Edges: Flow < Capacity
Nodes: Flow in = Flow out

Min-cut/Max-flow Theorem
In every network, the maximum flow equals the cost of the st-mincut

Slide credit: Pushmeet Kohli
## History of Maxflow Algorithms

### Augmenting Path and Push-Relabel

<table>
<thead>
<tr>
<th>Year</th>
<th>Discoverer(s)</th>
<th>Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>1951</td>
<td>Dantzig</td>
<td>$O(n^2mU)$</td>
</tr>
<tr>
<td>1955</td>
<td>Ford &amp; Fulkerson</td>
<td>$O(m^2U)$</td>
</tr>
<tr>
<td>1970</td>
<td>Dinitz</td>
<td>$O(n^2m)$</td>
</tr>
<tr>
<td>1972</td>
<td>Edmonds &amp; Karp</td>
<td>$O(m^2 \log U)$</td>
</tr>
<tr>
<td>1973</td>
<td>Dinitz</td>
<td>$O(nm \log U)$</td>
</tr>
<tr>
<td>1974</td>
<td>Karzanov</td>
<td>$O(n^3)$</td>
</tr>
<tr>
<td>1977</td>
<td>Cherkassky</td>
<td>$O(n^2m^{1/2})$</td>
</tr>
<tr>
<td>1980</td>
<td>Galil &amp; Naamad</td>
<td>$O(nm \log^2 n)$</td>
</tr>
<tr>
<td>1983</td>
<td>Sleator &amp; Tarjan</td>
<td>$O(nm \log n)$</td>
</tr>
<tr>
<td>1986</td>
<td>Goldberg &amp; Tarjan</td>
<td>$O(nm \log(n^2/m))$</td>
</tr>
<tr>
<td>1987</td>
<td>Ahuja &amp; Orlin</td>
<td>$O(nm + n^2 \log U)$</td>
</tr>
<tr>
<td>1987</td>
<td>Ahuja et al.</td>
<td>$O(nm \log(n\sqrt{\log U/m}))$</td>
</tr>
<tr>
<td>1989</td>
<td>Cheriyan &amp; Hagerup</td>
<td>$E(nm + n^2 \log^2 n)$</td>
</tr>
<tr>
<td>1990</td>
<td>Cheriyan et al.</td>
<td>$O(n^3/\log n)$</td>
</tr>
<tr>
<td>1990</td>
<td>Alon</td>
<td>$O(nm + n^{8/3} \log n)$</td>
</tr>
<tr>
<td>1992</td>
<td>King et al.</td>
<td>$O(nm + n^{2+\epsilon})$</td>
</tr>
<tr>
<td>1993</td>
<td>Phillips &amp; Westbrook</td>
<td>$O(nm(\log_{m/n} n + \log^{2+\epsilon} n))$</td>
</tr>
<tr>
<td>1994</td>
<td>King et al.</td>
<td>$O(nm \log_{m/(n \log n)} n)$</td>
</tr>
<tr>
<td>1997</td>
<td>Goldberg &amp; Rao</td>
<td>$O(n^{3/2} \log(n^2/m) \log U)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$O(n^{2/3} m \log(n^2/m) \log U)$</td>
</tr>
</tbody>
</table>

$n$: #nodes  
$m$: #edges  
$U$: maximum edge weight  

Algorithms assume non-negative edge weights

Slide credit: Andrew Goldberg
Maxflow Algorithms

Augmenting Path Based Algorithms

1. Find path from source to sink with positive capacity
2. Push maximum possible flow through this path
3. Repeat until no path can be found

Algorithms assume non-negative capacity

Slide credit: Pushmeet Kohli
Maxflow Algorithms

Flow = 0

Augmenting Path Based Algorithms

1. Find path from source to sink with positive capacity
2. Push maximum possible flow through this path
3. Repeat until no path can be found

Algorithms assume non-negative capacity

Slide credit: Pushmeet Kohli
Maxflow Algorithms

Flow = 0 + 2

Augmenting Path Based Algorithms

1. Find path from source to sink with positive capacity

2. Push maximum possible flow through this path

3. Repeat until no path can be found

Algorithms assume non-negative capacity

Slide credit: Pushmeet Kohli
Maxflow Algorithms

Augmenting Path Based Algorithms

1. Find path from source to sink with positive capacity
2. Push maximum possible flow through this path
3. Repeat until no path can be found

Algorithms assume non-negative capacity

Slide credit: Pushmeet Kohli
Maxflow Algorithms

Augmenting Path Based Algorithms

1. Find path from source to sink with positive capacity
2. Push maximum possible flow through this path
3. Repeat until no path can be found

Algorithms assume non-negative capacity
Maxflow Algorithms

Flow = 2

Augmenting Path Based Algorithms

1. Find path from source to sink with positive capacity
2. Push maximum possible flow through this path
3. Repeat until no path can be found

Algorithms assume non-negative capacity
Maxflow Algorithms

Augmenting Path Based Algorithms

1. Find path from source to sink with positive capacity
2. Push maximum possible flow through this path
3. Repeat until no path can be found

Algorithms assume non-negative capacity

Flow = 2 + 4
Maxflow Algorithms

Augmenting Path Based Algorithms

1. Find path from source to sink with positive capacity
2. Push maximum possible flow through this path
3. Repeat until no path can be found

Algorithms assume non-negative capacity

Slide credit: Pushmeet Kohli
Maxflow Algorithms

Augmenting Path Based Algorithms

1. Find path from source to sink with positive capacity
2. Push maximum possible flow through this path
3. Repeat until no path can be found

Algorithms assume non-negative capacity

Flow = 6
Maxflow Algorithms

Flow = 6 + 1

Augmenting Path Based Algorithms

1. Find path from source to sink with positive capacity
2. Push maximum possible flow through this path
3. Repeat until no path can be found

Algorithms assume non-negative capacity

Slide credit: Pushmeet Kohli
Maxflow Algorithms

Augmenting Path Based Algorithms

1. Find path from source to sink with positive capacity
2. Push maximum possible flow through this path
3. Repeat until no path can be found

Algorithms assume non-negative capacity

Flow = 7
Maxflow Algorithms

Augmenting Path Based Algorithms

1. Find path from source to sink with positive capacity

2. Push maximum possible flow through this path

3. Repeat until no path can be found

Algorithms assume non-negative capacity
Applications: Maxflow in Computer Vision

- Specialized algorithms for vision problems
  - Grid graphs
  - Low connectivity (m ~ O(n))

- Dual search tree augmenting path algorithm
  [Boykov and Kolmogorov PAMI 2004]
  - Finds approximate shortest augmenting paths efficiently.
  - High worst-case time complexity.
  - Empirically outperforms other algorithms on vision problems.
  - Efficient code available on the web
    http://www.cs.ucl.ac.uk/staff/V.Kolmogorov/software.html

Slide credit: Pushmeet Kohli
When Can s-t Graph Cuts Be Applied?

Regional term \[ E(L) = \sum_{p} E_p(L_p) \]

Boundary term \[ + \sum_{pq \in N} E(L_p, L_q) \]

- **s-t graph cuts** can only globally minimize **binary energies** that are **submodular**.  
  
  \[ E(L) \text{ can be minimized by s-t graph cuts} \iff E(s,s) + E(t,t) \leq E(s,t) + E(t,s) \]

  \[ \text{Submodularity ("convexity")} \]

- Non-submodular cases can still be addressed with some optimality guarantees.
  
  - Current research topic

Topics of This Lecture

- Segmentation as Energy Minimization
  - Markov Random Fields
  - Energy formulation

- Graph cuts for image segmentation
  - Basic idea
  - s-t Mincut algorithm
  - Extension to non-binary case

- Applications
  - Interactive segmentation
Dealing with Non-Binary Cases

- Limitation to binary energies is often a nuisance.
  ⇒ E.g. binary segmentation only...
- We would like to solve also multi-label problems.
  - The bad news: Problem is NP-hard with 3 or more labels!
- There exist some approximation algorithms which extend graph cuts to the multi-label case:
  - $\alpha$-Expansion
  - $\alpha\beta$-Swap
- They are no longer guaranteed to return the globally optimal result.
  - But $\alpha$-Expansion has a guaranteed approximation quality (2-approx) and converges in a few iterations.
**α-Expansion Move**

- **Basic idea:**
  - Break multi-way cut computation into a sequence of binary s-t cuts.

Slide credit: Yuri Boykov
\(\alpha\)-Expansion Algorithm

1. Start with any initial solution
2. For each label “\(\alpha\)” in any (e.g. random) order:
   1. Compute optimal \(\alpha\)-expansion move (s-t graph cuts).
   2. Decline the move if there is no energy decrease.
3. Stop when no expansion move would decrease energy.
Example: Stereo Vision

Original pair of “stereo” images

Depth map

ground truth

Slide credit: Yuri Boykov
**α-Expansion Moves**

- In each $α$-expansion a given label “$α$” grabs space from other labels

For each move, we choose the expansion that gives the largest decrease in the energy: $\Rightarrow$ binary optimization problem

Slide credit: Yuri Boykov
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GraphCut Applications: “GrabCut”

- Interactive Image Segmentation [Boykov & Jolly, ICCV’01]
  - Rough region cues sufficient
  - Segmentation boundary can be extracted from edges

- Procedure
  - User marks foreground and background regions with a brush.
  - This is used to create an initial segmentation which can then be corrected by additional brush strokes.

User segmentation cues

Additional segmentation cues

Slide credit: Matthieu Bray
**GrabCut: Data Model**

- Obtained from interactive user input
  - User marks foreground and background regions with a brush
  - Alternatively, user can specify a bounding box

Global optimum of the energy

---

Slide credit: Carsten Rother
GrabCut: Coherence Model

- An object is a coherent set of pixels:

\[
\psi(x, y) = \gamma \sum_{(m,n) \in C} \delta[x_n \neq x_m] e^{-\beta \|y_m - y_n\|^2}
\]

How to choose \(\gamma\)?

Error (%) over training set:
Iterated Graph Cuts

Result

Color model (Mixture of Gaussians)

Energy after each iteration

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Slide credit: Carsten Rother
GrabCut: Example Results

- This is included in the newest version of MS Office!

Image source: Carsten Rother
Applications: Interactive 3D Segmentation
Summary: Graph Cuts Segmentation

• **Pros**
  - Powerful technique, based on probabilistic model (MRF).
  - Applicable for a wide range of problems.
  - Very efficient algorithms available for vision problems.
  - Becoming a de-facto standard for many segmentation tasks.

• **Cons/Issues**
  - Graph cuts can only solve a limited class of models
    - Submodular energy functions
    - Can capture only part of the expressiveness of MRFs
  - Only approximate algorithms available for multi-label case
References and Further Reading

• A gentle introduction to Graph Cuts can be found in the following paper:

• Read how the interactive segmentation is realized in MS Office 2010:

• Try the GraphCut implementation at
  [http://www.cs.ucl.ac.uk/staff/V.Kolmogorov/software.html](http://www.cs.ucl.ac.uk/staff/V.Kolmogorov/software.html)