Machine Learning - Lecture 5

Linear Discriminants 2

28.04.2011

Bastian Leibe
RWTH Aachen
http://www.mmp.rwth-aachen.de

leibe@umic.rwth-aachen.de
Course Outline

- **Fundamentals (2 weeks)**
  - Bayes Decision Theory
  - Probability Density Estimation

- **Discriminative Approaches (5 weeks)**
  - Linear Discriminant Functions
  - Support Vector Machines
  - Ensemble Methods & Boosting
  - Randomized Trees, Forests & Ferns

- **Generative Models (4 weeks)**
  - Bayesian Networks
  - Markov Random Fields
Recap: Linear Discriminant Functions

- **Basic idea**
  - Directly encode decision boundary
  - Minimize misclassification probability directly.

- **Linear discriminant functions**
  \[ y(x) = w^T x + w_0 \]
  - \( w, w_0 \) define a hyperplane in \( \mathbb{R}^D \).
  - If a data set can be perfectly classified by a linear discriminant, then we call it **linearly separable**.
Recap: Least-Squares Classification

• Simplest approach
  - Directly try to minimize the sum-of-squares error
    \[ E(w) = \sum_{n=1}^{N} (y(x_n; w) - t_n)^2 \]
    \[ E_D(\mathbf{\tilde{W}}) = \frac{1}{2} \text{Tr} \left\{ (\mathbf{\tilde{X}\tilde{W}} - \mathbf{T})^T (\mathbf{\tilde{X}\tilde{W}} - \mathbf{T}) \right\} \]
  - Setting the derivative to zero yields
    \[ \mathbf{\tilde{W}} = (\mathbf{\tilde{X}^T\tilde{X}})^{-1} \mathbf{\tilde{X}^T} \mathbf{T} = \mathbf{\tilde{X}}^\dagger \mathbf{T} = (\mathbf{\tilde{X}^T\tilde{X}})^{-1} \mathbf{\tilde{X}^T} \mathbf{T} \]
  - We then obtain the discriminant function as
    \[ y(x) = \mathbf{\tilde{W}^T} \mathbf{\tilde{x}} = \mathbf{T}^T \left( \mathbf{\tilde{X}}^\dagger \right)^T \mathbf{\tilde{x}} \]
  - \(\Rightarrow\) Exact, closed-form solution for the discriminant function parameters.
Recap: Problems with Least Squares

- Least-squares is very sensitive to outliers!
  - The error function penalizes predictions that are “too correct”.

Image source: C.M. Bishop, 2006
Recap: Generalized Linear Models

- Generalized linear model
  \[ y(x) = g(w^T x + w_0) \]
  - \( g(\cdot) \) is called an activation function and may be nonlinear.
  - The decision surfaces correspond to
    \[ y(x) = \text{const.} \iff w^T x + w_0 = \text{const}. \]
  - If \( g \) is monotonous (which is typically the case), the resulting decision boundaries are still linear functions of \( x \).

- Advantages of the non-linearity
  - Can be used to bound the influence of outliers and “too correct” data points.
  - When using a sigmoid for \( g(\cdot) \), we can interpret the \( y(x) \) as posterior probabilities.

\[
g(a) \equiv \frac{1}{1 + \exp(-a)}
\]
Recap: Linear Separability

- Up to now: restrictive assumption
  - Only consider linear decision boundaries

- Classical counterexample: XOR
Recap: Extension to Nonlinear Basis Fcts.

- **Generalization**
  - Transform vector $x$ with $M$ nonlinear basis functions $\phi_j(x)$:
    \[
    y_k(x) = \sum_{j=1}^{M} w_{kj} \phi_j(x) + w_{k0}
    \]

- **Advantages**
  - Transformation allows non-linear decision boundaries.
  - By choosing the right $\phi_j$, every continuous function can (in principle) be approximated with arbitrary accuracy.

- **Disadvantage**
  - The error function can in general no longer be minimized in closed form.
    \[\Rightarrow\] Minimization with Gradient Descent
Recap: Gradient Descent

• Iterative minimization
  - Start with an initial guess for the parameter values $w_{k,j}^{(0)}$.
  - Move towards a (local) minimum by following the gradient.

• Basic strategies
  - “Batch learning”
    $$w_{k,j}^{(\tau+1)} = w_{k,j}^{(\tau)} - \eta \left. \frac{\partial E(w)}{\partial w_{k,j}} \right|_{w^{(\tau)}}$$

  - “Sequential updating”
    $$w_{k,j}^{(\tau+1)} = w_{k,j}^{(\tau)} - \eta \left. \frac{\partial E_n(w)}{\partial w_{k,j}} \right|_{w^{(\tau)}}$$

where
$$E(w) = \sum_{n=1}^{N} E_n(w)$$

B. Leibe
Recap: Gradient Descent

- Example: Quadratic error function
  \[ E(w) = \sum_{n=1}^{N} (y(x_n; w) - t_n)^2 \]

- Sequential updating leads to delta rule (=LMS rule)
  \[ w_{kj}^{(\tau+1)} = w_{kj}^{(\tau)} - \eta (y_k(x_n; w) - t_{kn}) \phi_j(x_n) \]
  \[ = w_{kj}^{(\tau)} - \eta \delta_{kn} \phi_j(x_n) \]
  - where
    \[ \delta_{kn} = y_k(x_n; w) - t_{kn} \]

⇒ Simply feed back the input data point, weighted by the classification error.
Recap: Gradient Descent

- Cases with differentiable, non-linear activation function

\[ y_k(x) = g(a_k) = g \left( \sum_{j=0}^{M} w_{k} \phi_j(x_n) \right) \]

- Gradient descent (again with quadratic error function)

\[
\begin{align*}
\frac{\partial E_n(w)}{\partial w_{k,j}} &= \frac{\partial g(a_k)}{\partial w_{k,j}} \left( y_k(x_n; w) - t_{kn} \right) \phi_j(x_n) \\

w_{k,j}^{(\tau+1)} &= w_{k,j}^{(\tau)} - \eta \delta_{kn} \phi_j(x_n) \\

\delta_{kn} &= \frac{\partial g(a_k)}{\partial w_{k,j}} \left( y_k(x_n; w) - t_{kn} \right)
\end{align*}
\]
Topics of This Lecture

• Fisher’s linear discriminant (FLD)
  - Classification as dimensionality reduction
  - Linear discriminant analysis
  - Multiple discriminant analysis
  - Applications

• Logistic Regression
  - Probabilistic discriminative models
  - Logistic sigmoid (logit function)
  - Cross-entropy error
  - Gradient descent
  - Iteratively Reweighted Least Squares

• Note on Error Functions
Classification as Dimensionality Reduction

- Classification as dimensionality reduction
  - We can interpret the linear classification model as a projection onto a lower-dimensional space.
  - E.g., take the $D$-dimensional input vector $\mathbf{x}$ and project it down to one dimension by applying the function
    
    $$y = \mathbf{w}^T \mathbf{x}$$
  - If we now place a threshold at $y \geq -w_0$, we obtain our standard two-class linear classifier.
  - The classifier will have a lower error the better this projection separates the two classes.

⇒ New interpretation of the learning problem
  - Try to find the projection vector $\mathbf{w}$ that maximizes the class separation.
Classification as Dimensionality Reduction

- Two questions
  - How to measure class separation?
  - How to find the best projection (with maximal class separation)?
Classification as Dimensionality Reduction

- Measuring class separation
  - We could simply measure the separation of the class means.
  - Choose \( w \) so as to maximize
    \[
    (m_2 - m_1) = w^T (m_2 - m_1)
    \]

- Problems with this approach
  1. This expression can be made arbitrarily large by increasing \( \|w\| \).
     ⇒ Need to enforce additional constraint \( \|w\| = 1 \).
     ⇒ This constrained minimization results in
        \[
        w \propto (m_2 - m_1)
        \]
  2. The criterion may result in bad separation if the clusters have elongated shapes.
Fisher’s Linear Discriminatnt Analysis (FLD)

- Better idea:
  - Find a projection that maximizes the ratio of the between-class variance to the within-class variance:
    \[
    J(w) = \frac{(m_2 - m_1)^2}{s_1^2 + s_2^2}
    \]
    with
    \[
    s_k^2 = \sum_{n \in C_k} (y_n - m_k)^2
    \]
  - Usually, this is written as
    \[
    J(w) = \frac{w^T S_B w}{w^T S_W w}
    \]
  - Where
    \[
    S_B = (m_2 - m_1)(m_2 - m_1)^T
    \]
    \[
    S_W = \sum_{k=1}^{2} \sum_{n \in C_k} (x_n - m_k)(x_n - m_k)^T
    \]
Fisher’s Linear Discriminant Analysis (FLD)

- Maximize distance between classes
- Minimize distance within a class
- Criterion: \( J(w) = \frac{w^T S_B w}{w^T S_W w} \)
  - \( S_B \) ... between-class scatter matrix
  - \( S_W \) ... within-class scatter matrix
- The optimal solution for \( w \) can be obtained as:
  \[ w \propto S_W^{-1}(m_2 - m_1) \]
- Classification function:
  \[ y(x) = w^T x + w_0 \]
  where \( w_0 = -w^T m \)
Multiple Discriminant Analysis

- Generalization to \( K \) classes

\[
J(W) = \frac{|W^T S_B W|}{|W^T S_W W|}
\]

where

\[
W = [w_1, \ldots, w_K] \quad m = \frac{1}{N} \sum_{n=1}^{N} x_n = \frac{1}{N} \sum_{k=1}^{K} N_k m_k
\]

\[
S_B = \sum_{k=1}^{K} N_k (m_k - m)(m_k - m)^T
\]

\[
S_W = \sum_{k=1}^{K} \sum_{n \in C_k} (x_n - m_k)(x_n - m_k)^T
\]

B. Leibe
Maximizing $J(W)$

- Generalized eigenvalue problem
  
  $$J(W) = \frac{|W^T S_B W|}{|W^T S_W W|}$$

  - The columns of the optimal $W$ are the eigenvectors corresponding to the largest eigenvalues of
    $$S_B w_i = \lambda_i S_W w_i$$

  - Defining $v = S_B^{\frac{1}{2}} w$, we get
    $$S_B^{\frac{1}{2}} S_W^{-1} S_B^{\frac{1}{2}} v = \lambda v$$

    which is a regular eigenvalue problem.

    $\Rightarrow$ Solve to get eigenvectors of $v$, then from that of $w$.

- For the K-class case we obtain (at most) $K-1$ projections.
  - (i.e. eigenvectors corresponding to non-zero eigenvalues.)
What Does It Mean?

- What does it mean to apply a linear classifier?

\[
y(x) = \tilde{w}^T \tilde{x}
\]

Weight vector \hspace{1cm} Input vector

- Classifier interpretation

  - The weight vector has the same dimensionality as \( x \).
  - Positive contributions where \( \text{sign}(x_i) = \text{sign}(w_i) \).
    \( \Rightarrow \) The weight vector identifies which input dimensions are important for positive or negative classification (large \( |w_i| \)) and which ones are irrelevant (near-zero \( w_i \)).
    \( \Rightarrow \) If the inputs \( x \) are normalized, we can interpret \( w \) as a “template” vector that the classifier tries to match.

\[
w^T x = \|w\| \|x\| \cos \theta
\]

B. Leibe
Example Application: Fisherfaces

- Visual discrimination task
  - Training data:
    - $C_1$: Subjects with glasses
    - $C_2$: Subjects without glasses

- Test:
  - glasses? Take each image as a vector of pixel values and apply FLD...

*Image source: Yale Face Database*
Fisherfaces: Interpretability

- Resulting weight vector for “Glasses/NoGlasses“
Summary: Fisher’s Linear Discriminant

- **Properties**
  - Simple method for dimensionality reduction, preserves class discriminability.
  - Can use parametric methods in reduced-dim. space that might not be feasible in original higher-dim. space.
  - Widely used in practical applications.

- **Restrictions / Caveats**
  - Not possible to get more than \( K-1 \) projections.
  - FLD reduces the computation to class means and covariances.
  - Implicit assumption that class distributions are unimodal and well-approximated by a Gaussian/hyperellipsoid.
Topics of This Lecture

- Fisher’s linear discriminant (FLD)
  - Classification as dimensionality reduction
  - Linear discriminant analysis
  - Multiple discriminant analysis
  - Applications

- Logistic Regression
  - Probabilistic discriminative models
  - Logistic sigmoid (logit function)
  - Cross-entropy error
  - Gradient descent
  - Iteratively Reweighted Least Squares

- Note on Error Functions
Probabilistic Discriminative Models

• We have seen that we can write

\[ p(C_1|x) = \sigma(a) \]

\[ = \frac{1}{1 + \exp(-a)} \]

• We can obtain the familiar probabilistic model by setting

\[ a = \ln \frac{p(x|C_1)p(C_1)}{p(x|C_2)p(C_2)} \]

• Or we can use generalized linear discriminant models

\[ a = w^T x \]

or \[ a = w^T \phi(x) \]
Probabilistic Discriminative Models

- In the following, we will consider models of the form

\[ p(C_1|\phi) = y(\phi) = \sigma(w^T \phi) \]

with

\[ p(C_2|\phi) = 1 - p(C_1|\phi) \]

- This model is called logistic regression.

- Why should we do this? What advantage does such a model have compared to modeling the probabilities?

\[ p(C_1|\phi) = \frac{p(\phi|C_1)p(C_1)}{p(\phi|C_1)p(C_1) + p(\phi|C_2)p(C_2)} \]

- Any ideas?
Comparison

Let’s look at the number of parameters...

- Assume we have an $M$-dimensional feature space $\phi$.
- And assume we represent $p(\phi | C_k)$ and $p(C_k)$ by Gaussians.
- How many parameters do we need?
  - For the means: $2M$
  - For the covariances: $M(M+1)/2$
  - Together with the class priors, this gives $M(M+5)/2+1$ parameters!

- How many parameters do we need for logistic regression?
  \[ p(C_1 | \phi) = y(\phi) = \sigma(w^T \phi) \]
  - Just the values of $w \Rightarrow M$ parameters.

$\Rightarrow$ For large $M$, logistic regression has clear advantages!
Logistic Sigmoid

- **Properties**
  - **Definition:** \( \sigma(a) = \frac{1}{1 + \exp(-a)} \)
  - **Inverse:** \( a = \ln \left( \frac{\sigma}{1 - \sigma} \right) \) “logit” function
  - **Symmetry property:** \( \sigma(-a) = 1 - \sigma(a) \)
  - **Derivative:** \( \frac{d\sigma}{da} = \sigma(1 - \sigma) \)
Logistic Regression

- Let’s consider a data set \( \{ \phi_n, t_n \} \) with \( n = 1, \ldots, N \), where \( \phi_n = \phi(x_n) \) and \( t_n \in \{0, 1\} \), \( t = (t_1, \ldots, t_N)^T \).

- With \( y_n = p(C_1|\phi_n) \), we can write the likelihood as

\[
p(t|w) = \prod_{n=1}^{N} y_n^{t_n} \{1 - y_n\}^{1-t_n}
\]

- Define the error function as the negative log-likelihood

\[
E(w) = -\ln p(t|w)
\]

\[
= - \sum_{n=1}^{N} \{t_n \ln y_n + (1 - t_n) \ln(1 - y_n)\}
\]

- This is the so-called cross-entropy error function.
Gradient of the Error Function

- **Error function**

\[ E(w) = - \sum_{n=1}^{N} \{ t_n \ln y_n + (1 - t_n) \ln(1 - y_n) \} \]

- **Gradient**

\[ \nabla E(w) = - \sum_{n=1}^{N} \left\{ t_n \frac{d}{dw} \frac{y_n}{y_n} + (1 - t_n) \frac{d}{dw} \frac{(1 - y_n)}{(1 - y_n)} \right\} \]

\[ = - \sum_{n=1}^{N} \left\{ t_n \frac{y_n (1 - y_n)}{y_n} \phi_n - (1 - t_n) \frac{y_n (1 - y_n)}{(1 - y_n)} \phi_n \right\} \]

\[ = - \sum_{n=1}^{N} \left\{ (t_n - t_n y_n - y_n + t_n y_n) \phi_n \right\} \]

\[ = \sum_{n=1}^{N} (y_n - t_n) \phi_n \]
Gradient of the Error Function

- Gradient for logistic regression

\[ \nabla E(w) = \sum_{n=1}^{N} (y_n - t_n) \phi_n \]

- Does this look familiar to you?

- This is the same result as for the Delta (\(=\)LMS) rule

\[ w_{k,j}^{(\tau+1)} = w_{k,j}^{(\tau)} - \eta (y_k(x_n; w) - t_{kn}) \phi_j(x_n) \]

- We can use this to derive a sequential estimation algorithm.
  - However, this will be quite slow...
A More Efficient Iterative Method...

- Second-order Newton-Raphson gradient descent scheme
  \[ w^{(\tau+1)} = w^{(\tau)} - H^{-1} \nabla E(w) \]

  where \( H = \nabla \nabla E(w) \) is the Hessian matrix, i.e. the matrix of second derivatives.

- Properties
  - Local quadratic approximation to the log-likelihood.
  - Faster convergence.
Newton-Raphson for Least-Squares Estimation

• Let’s first apply Newton-Raphson to the least-squares error function:

\[
E(w) = \frac{1}{2} \sum_{n=1}^{N} (w^T \phi_n - t_n)^2
\]

\[
\nabla E(w) = \sum_{n=1}^{N} (w^T \phi_n - t_n) \phi_n = \Phi^T \Phi w - \Phi^T t
\]

\[
H = \nabla \nabla E(w) = \sum_{n=1}^{N} \phi_n \phi_n^T = \Phi^T \Phi
\]

where \( \Phi = \begin{bmatrix} \phi_1^T \\ \vdots \\ \phi_N^T \end{bmatrix} \)

• Resulting update scheme:

\[
w^{(\tau+1)} = w^{(\tau)} - (\Phi^T \Phi)^{-1} (\Phi^T \Phi w^{(\tau)} - \Phi^T t)
\]

\[
= (\Phi^T \Phi)^{-1} \Phi^T t
\]

Closed-form solution!
Newton-Raphson for Logistic Regression

• Now, let’s try Newton-Raphson on the cross-entropy error function:

\[
E(w) = - \sum_{n=1}^{N} \{ t_n \ln y_n + (1 - t_n) \ln(1 - y_n) \}
\]

\[
\nabla E(w) = \sum_{n=1}^{N} (y_n - t_n) \phi_n = \Phi^T(y - t)
\]

\[
H = \nabla \nabla E(w) = \sum_{n=1}^{N} y_n (1 - y_n) \phi_n \phi_n^T = \Phi^T R \Phi
\]

where \( R \) is an \( N \times N \) diagonal matrix with \( R_{nn} = y_n(1 - y_n) \).

⇒ The Hessian is no longer constant, but depends on \( w \) through the weighting matrix \( R \).
Iteratively Reweighted Least Squares

- Update equations

\[ w^{(\tau+1)} = w^{(\tau)} - (\Phi^T R \Phi)^{-1} \Phi^T (y - t) \]

\[ = (\Phi^T R \Phi)^{-1} \left\{ \Phi^T R \Phi w^{(\tau)} - \Phi^T (y - t) \right\} \]

\[ = (\Phi^T R \Phi)^{-1} \Phi^T R z \]

with \[ z = \Phi w^{(\tau)} - R^{-1} (y - t) \]

- Again very similar form (normal equations)
  - But now with non-constant weighing matrix \( R \) (depends on \( w \)).
  - Need to apply normal equations iteratively.
  \[ \Rightarrow \text{Iteratively Reweighted Least-Squares (IRLS)} \]
Summary: Logistic Regression

- **Properties**
  - Directly represent posterior distribution $p(\phi | C_k)$
  - Requires fewer parameters than modeling the likelihood + prior.
  - Very often used in statistics.
  - It can be shown that the cross-entropy error function is concave
    - Optimization leads to unique minimum
    - But no closed-form solution exists
    - Iterative optimization (IRLS)
  - Both online and batch optimizations exist
  - There is a multi-class version described in (Bishop Ch.4.3.4).

- **Caveat**
  - Logistic regression tends to systematically overestimate odds ratios when the sample size is less than $\sim500$. 
Topics of This Lecture

- Fisher’s linear discriminant (FLD)
  - Classification as dimensionality reduction
  - Linear discriminant analysis
  - Multiple discriminant analysis
  - Applications

- Logistic Regression
  - Probabilistic discriminative models
  - Logistic sigmoid (logit function)
  - Cross-entropy error
  - Gradient descent
  - Iteratively Reweighted Least Squares

- Note on Error Functions

B. Leibe
We have now seen already a number of error functions:

- Ideal misclassification error
- Quadratic error
- Cross-entropy error
Error Functions

- **Ideal Misclassification Error**
  - This is what we would like to optimize.
  - But cannot compute gradients here.

- **Quadratic Error**
  - Easy to optimize, closed-form solutions exist.
  - But not robust to outliers.

- **Cross-Entropy Error**
  - Minimizer of this error is given by posterior class probabilities.
  - Concave error function, unique minimum exists.
  - But no closed-form solution, requires iterative estimation.
References and Further Reading

- More information on Linear Discriminant Functions can be found in Chapter 4 of Bishop’s book (in particular Chapter 4.1 - 4.3).

Christopher M. Bishop
Pattern Recognition and Machine Learning
Springer, 2006