Machine Learning - Lecture 7
Model Combination & Boosting

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Many slides adapted from B. Schiele
Course Outline

- **Fundamentals (2 weeks)**
  - Bayes Decision Theory
  - Probability Density Estimation

- **Discriminative Approaches (4 weeks)**
  - Linear Discriminant Functions
  - Statistical Learning Theory & SVMs
  - Ensemble Methods & Boosting
  - Randomized Trees, Forests & Ferns

- **Generative Models (4 weeks)**
  - Bayesian Networks
  - Markov Random Fields

- **Unifying Perspective (2 weeks)**

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Recap: SVM for Non-Separable Data

- Slack variables
  - One slack variable $\xi_n \geq 0$ for each training data point.

- Interpretation
  - $\xi_n = 0$ for points that are on the correct side of the margin.
  - $\xi_n = |t_n - y(x_n)|$ for all other points.

- We do not have to set the slack variables ourselves!
  $\Rightarrow$ They are jointly optimized together with $w$. 

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Recap: SVM - New Dual Formulation

• New SVM Dual: Maximize

\[ L_d(a) = \sum_{n=1}^{N} a_n - \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m t_n t_m (x_m^T x_n) \]

under the conditions

\[ 0 \leq a_n \leq C \]

\[ \sum_{n=1}^{N} a_n t_n = 0 \]

• This is again a quadratic programming problem

⇒ Solve as before...

This is all that changed!
Recap: Nonlinear SVMs

- General idea: The original input space can be mapped to some higher-dimensional feature space where the training set is separable:

\[ \Phi: x \rightarrow \phi(x) \]
Recap: The Kernel Trick

- Important observation
  - \( \phi(x) \) only appears in the form of dot products \( \phi(x)^T \phi(y) \):
    \[
    y(x) = w^T \phi(x) + b
    \]
    \[
    = \sum_{n=1}^{N} a_n t_n \phi(x_n)^T \phi(x) + b
    \]
  - Define a so-called kernel function \( k(x,y) = \phi(x)^T \phi(y) \).
  - Now, in place of the dot product, use the kernel instead:
    \[
    y(x) = \sum_{n=1}^{N} a_n t_n k(x_n, x) + b
    \]
  - The kernel function \textit{implicitly} maps the data to the higher-dimensional space (without having to compute \( \phi(x) \) explicitly)!
Recap: Kernels Fulfiling Mercer’s Condition

- **Polynomial kernel**
  \[ k(x, y) = (x^T y + 1)^p \]

- **Radial Basis Function kernel**
  \[ k(x, y) = \exp \left\{ -\frac{(x - y)^2}{2\sigma^2} \right\} \]  
  e.g. Gaussian

- **Hyperbolic tangent kernel**
  \[ k(x, y) = \tanh(\kappa x^T y + \delta) \]  
  e.g. Sigmoid

(and many, many more...)
Recap: Nonlinear SVM - Dual Formulation

- **SVM Dual: Maximize**

\[
L_d(a) = \sum_{n=1}^{N} a_n - \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m t_n t_m k(x_m, x_n)
\]

under the conditions

\[
0 \leq a_n \leq C
\]

\[
\sum_{n=1}^{N} a_n t_n = 0
\]

- **Classify new data points using**

\[
y(x) = \sum_{n=1}^{N} a_n t_n k(x_n, x) + b
\]

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So Far...

- We’ve seen already a variety of different classifiers
  - k-NN
  - Bayes classifiers
  - Linear discriminants
  - SVMs

- Each of them has their strengths and weaknesses...
  - Can we improve performance by combining them?
Topics of This Lecture

• Ensembles of Classifiers

• Constructing Ensembles
  - Cross-validation
  - Bagging

• Combining Classifiers
  - Stacking
  - Bayesian model averaging
  - Boosting

• AdaBoost
  - Intuition
  - Algorithm
  - Analysis
  - Extensions

• Applications
Ensembles of Classifiers

• Intuition
  - Assume we have $K$ classifiers.
  - They are independent (i.e. their errors are uncorrelated).
  - Each of them has an error probability $p < 0.5$ on training data.
    - Why can we assume that $p$ won’t be larger than 0.5?
  - Then a simple majority vote of all classifiers should have a lower error than each individual classifier...
Ensembles of Classifiers

• Example
  - \( K \) classifiers with error probability \( p = 0.3 \).
  - Probability that exactly \( L \) classifiers make an error:
    \[
    p^L (1 - p)^{K-L}
    \]
  - The probability that 11 or more classifiers make an error is 0.026.
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  - Analysis
  - Extensions

Methods for obtaining a set of classifiers

Methods for combining different classifiers

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Constructing Ensembles

• How do we get different classifiers?
  - Simplest case: train same classifier on different data.
  - But... where shall we get this additional data from?
    - Recall: training data is very expensive!

• Idea: Subsample the training data
  - Reuse the same training algorithm several times on different subsets of the training data.

• Well-suited for “unstable” learning algorithms
  - Unstable: small differences in training data can produce very different classifiers
    - E.g. Decision trees, neural networks, rule learning algorithms,...
  - Stable learning algorithms
    - E.g. Nearest neighbor, linear regression, SVMs,...
Constructing Ensembles

• Cross-Validation
  - Split the available data into N disjunct subsets.
  - In each run, train on N-1 subsets for training a classifier.
  - Estimate the generalization error on the held-out validation set.

• E.g. 5-fold cross-validation

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## Constructing Ensembles

### Bagging = “Bootstrap aggregation” (Breiman 1996)

- In each run of the training algorithm, randomly select $M$ samples from the full set of $N$ training data points.
- If $M = N$, then on average, 63.2% of the training points will be represented. The rest are duplicates.

### Injecting randomness

- Many (iterative) learning algorithms need a random initialization (e.g. k-means, EM)
- Perform multiple runs of the learning algorithm with different random initializations.
Topics of This Lecture

- **Ensembles of Classifiers**
- **Constructing Ensembles**
  - Cross-validation
  - Bagging
- **Combining Classifiers**
  - Stacking
  - Bayesian Model Averaging
  - Boosting
- **AdaBoost**
  - Intuition
  - Algorithm
  - Analysis
  - Extensions
- **Applications**
Stacking

• Idea
  - Learn $L$ classifiers (based on the training data)
  - Find a meta-classifier that takes as input the output of the $L$ first-level classifiers.

• Example
  - Learn $L$ classifiers with leave-one-out.
  - Interpret the prediction of the $L$ classifiers as $L$-dimensional feature vector.
  - Learn “level-2” classifier based on the examples generated this way.
Stacking

• Idea
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  - Find a meta-classifier that takes as input the output of the $L$ first-level classifiers.

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  - Interpret the prediction of the $L$ classifiers as $L$-dimensional feature vector.
  - Learn “level-2” classifier based on the examples generated this way.
Stacking

- Why can this be useful?
  - Simplicity
    - We may already have several existing classifiers available.
      ⇒ No need to retrain those, they can just be combined with the rest.
  - Correlation between classifiers
    - The combination classifier can learn the correlation.
      ⇒ Better results than simple Naïve Bayes combination.
  - Feature combination
    - E.g. combine information from different sensors or sources
      (vision, audio, acceleration, temperature, radar, etc.).
    - We can get good training data for each sensor individually,
      but data from all sensors together is rare.
      ⇒ Train each of the L classifiers on its own input data.
      Only combination classifier needs to be trained on combined input.
Recap: Model Combination

- E.g. Mixture of Gaussians
  - Several components are combined probabilistically.
  - Interpretation: different data points can be generated by different components.
  - We model the uncertainty which mixture component is responsible for generating the corresponding data point:
    \[
    p(x) = \sum_{k=1}^{K} \pi_k \mathcal{N}(x|\mu_k, \Sigma_k)
    \]
  - For iid data, we write the marginal probability of a data set \(X = \{x_1, \ldots, x_N\}\) in the form:
    \[
    p(X) = \prod_{n=1}^{N} p(x_n) = \prod_{n=1}^{N} \sum_{k=1}^{K} \pi_k \mathcal{N}(x_n|\mu_k, \Sigma_k)
    \]
Bayesian Model Averaging

- **Model Averaging**
  - Suppose we have $H$ different models $h = 1, \ldots, H$ with prior probabilities $p(h)$.
  - Construct the marginal distribution over the data set
    \[
    p(X) = \sum_{h=1}^{H} p(X|h)p(h)
    \]

- **Interpretation**
  - Just one model is responsible for generating the entire data set.
  - The probability distribution over $h$ just reflects our uncertainty which model that is.
  - As the size of the data set increases, this uncertainty reduces, and $p(X|h)$ becomes focused on just one of the models.
Note the Different Interpretations!

- **Model Combination**
  - Different data points *generated by different model components*.
  - Uncertainty is about which component created which data point.
  ⇒ One latent variable $z_n$ for each data point:
  \[
  p(X) = \prod_{n=1}^{N} p(x_n) = \prod_{n=1}^{N} \sum_{z_n} p(x_n, z_n)
  \]

- **Bayesian Model Averaging**
  - The whole data set is *generated by a single model*.
  - Uncertainty is about which model was responsible.
  ⇒ One latent variable $z$ for the entire data set:
  \[
  p(X) = \sum_{z} p(X, z)
  \]
Model Averaging: Expected Error

- Combine $M$ predictors $y_m(x)$ for target output $h(x)$.
  - E.g. each trained on a different bootstrap data set by bagging.
  - The committee prediction is given by
    \[
    y_{COM}(x) = \frac{1}{M} \sum_{m=1}^{M} y_m(x)
    \]
  
- The output can be written as the true value plus some error.
  \[
  y(x) = h(x) + \epsilon(x)
  \]

- Thus, the average sum-of-squares error takes the form
  \[
  \mathbb{E}_x \left[ \left\{ y_m(x) - h(x) \right\}^2 \right] = \mathbb{E}_x \left[ \epsilon_m(x)^2 \right]
  \]
Model Averaging: Expected Error

- Average error of individual models
\[ E_{AV} = \frac{1}{M} \sum_{m=1}^{M} \mathbb{E}_x \left[ \epsilon_m(x)^2 \right] \]

- Average error of committee
\[ E_{COM} = \mathbb{E}_x \left[ \left\{ \frac{1}{M} \sum_{m=1}^{M} y_m(x) - h(x) \right\}^2 \right] = \mathbb{E}_x \left[ \left\{ \frac{1}{M} \sum_{m=1}^{M} \epsilon_m(x) \right\}^2 \right] \]

- Assumptions
  - Errors have zero mean: \( \mathbb{E}_x [\epsilon_m(x)] = 0 \)
  - Errors are uncorrelated: \( \mathbb{E}_x [\epsilon_m(x)\epsilon_j(x)] = 0 \)

- Then:
\[ E_{COM} = \frac{1}{M} E_{AV} \]

Isn’t this spectacular?

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Model Averaging: Expected Error

- Average error of committee

\[ E_{COM} = \frac{1}{M} E_{AV} \]

- This suggests that the average error of a model can be reduced by a factor of \( M \) simply by averaging \( M \) versions of the model!
- Spectacular indeed...
- This sounds almost too good to be true...

- And it is... Can you see where the problem is?
  - Unfortunately, this result depends on the assumption that the errors are all uncorrelated.
  - In practice, they will typically be highly correlated.
  - Still, it can be shown that

\[ E_{COM} \leq E_{AV} \]
Boosting

- Simple technique with very interesting properties
  - Combination of multiple classifiers with the goal to improve classification accuracy.
  - Can be used with many different types of classifiers.
  - None of them needs to be too good on its own.
    - In fact, they only have to be slightly better than chance.
    - Extreme case: Decision stumps

\[
y(x) = \begin{cases} 
1, & x_i \geq \theta \\
0, & \text{else}
\end{cases}
\]

- Main idea
  - Train successive component classifiers on a subset of the training data that is most informative given the current set of classifiers.

⇒ Sequential classifier selection
Boosting (Schapire 1989)

- Algorithm: (3-component classifier)
  1. Sample $N_1 < N$ training examples (without replacement) from training set $\mathcal{D}$ to get set $\mathcal{D}_1$.
     - Train weak classifier $C_1$ on $\mathcal{D}_1$.
  2. Sample $N_2 < N$ training examples (without replacement), half of which were misclassified by $C_1$ to get set $\mathcal{D}_2$.
     - Train weak classifier $C_2$ on $\mathcal{D}_2$.
  3. Choose all data in $\mathcal{D}$ on which $C_1$ and $C_2$ disagree to get set $\mathcal{D}_3$.
     - Train weak classifier $C_3$ on $\mathcal{D}_3$.
  4. Get the final classifier output by majority voting of $C_1$, $C_2$, and $C_3$.

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Image source: Duda, Hart, Stork, 2001
Applying Boosting

- How should we choose the number of samples $N_1$?
  - Ideally, the number of samples should be roughly equal in all 3 component classifiers.
  - Reasonable first guess: $N_1 \approx N/3$
  - However, if the problem is very simple
    - $C_1$ will explain most of the data.
      $\Rightarrow N_2$ and $N_3$ will be very small.
      $\Rightarrow$ Not all of the data will be used effectively.
  - Similarly, if the problem is extremely hard
    - $C_1$ will explain only a small part of the data.
      $\Rightarrow N_2$ may be unacceptably large.
  - In practice, may need to run the boosting procedure a few times and adjust $N_1$ in order to use the full training set.
  - Also, we can recursively apply the procedure on $C_1$ to $C_3$. 

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Discussion: Ensembles of Classifiers

• Set of simple methods for improving classification
  - Often effective in practice.

• Apparent contradiction
  - We have stressed before that a classifier should be trained on samples from the distribution on which it will be tested.
  - Resampling seems to violate this recommendation.
  - Why can a classifier trained on a weighted data distribution do better than one trained on the i.i.d. sample?

• Explanation
  - We do not attempt to model the full category distribution here.
  - Instead, try to find the decision boundary more directly.
  - Also, increasing number of component classifiers broadens the class of implementable decision functions.
Topics of This Lecture

• Ensembles of Classifiers
• Constructing Ensembles
  ➢ Cross-validation
  ➢ Bagging
• Combining Classifiers
  ➢ Stacking
  ➢ Bayesian model averaging
  ➢ Boosting

• AdaBoost
  ➢ Intuition
  ➢ Algorithm
  ➢ Analysis
  ➢ Extensions
• Applications
AdaBoost - “Adaptive Boosting”

• Main idea
  - Instead of resampling, reweight misclassified training examples.
    - Increase the chance of being selected in a sampled training set.
    - Or increase the misclassification cost when training on the full set.

• Components
  - \( h_m(x) \): “weak” or base classifier
    - Condition: <50% training error over any distribution
  - \( H(x) \): “strong” or final classifier

• AdaBoost:
  - Construct a strong classifier as a thresholded linear combination of the weighted weak classifiers:

\[
H(x) = \text{sign} \left( \sum_{m=1}^{M} \alpha_m h_m(x) \right)
\]
AdaBoost: Intuition

Consider a 2D feature space with **positive** and **negative** examples.

Each weak classifier splits the training examples with at least 50% accuracy.

Examples misclassified by a previous weak learner are given more emphasis at future rounds.

Slide credit: Kristen Grauman
AdaBoost: Intuition

Weak Classifier 1

Weights Increased

Weak Classifier 2

Figure adapted from Freund & Schapire
AdaBoost: Intuition

Final classifier is combination of the weak classifiers

Slide credit: Kristen Grauman

Figure adapted from Freund & Schapire
AdaBoost - Formalization

• 2-class classification problem
  - Given: training set \( X = \{ x_1, \ldots, x_N \} \)
    with target values \( T = \{ t_1, \ldots, t_N \} \), \( t_n \in \{-1,1\} \).
  - Associated weights \( W = \{ w_1, \ldots, w_N \} \) for each training point.

• Basic steps
  - In each iteration, AdaBoost trains a new weak classifier \( h_m(x) \)
    based on the current weighting coefficients \( W^{(m)} \).
  - We then adapt the weighting coefficients for each point
    - Increase \( w_n \) if \( x_n \) was misclassified by \( h_m(x) \).
    - Decrease \( w_n \) if \( x_n \) was classified correctly by \( h_m(x) \).
  - Make predictions using the final combined model
    \[
    H(x) = \text{sign} \left( \sum_{m=1}^{M} \alpha_m h_m(x) \right)
    \]
AdaBoost - Algorithm

1. Initialization: Set \( w_{n}^{(1)} = \frac{1}{N} \) for \( n = 1, \ldots, N \).

2. For \( m = 1, \ldots, M \) iterations
   a) Train a new weak classifier \( h_{m}(x) \) using the current weighting coefficients \( W^{(m)} \) by minimizing the weighted error function
      \[
      J_{m} = \sum_{n=1}^{N} w_{n}^{(m)} I(h_{m}(x) \neq t_{n})
      \]
   b) Estimate the weighted error of this classifier on \( X \):
      \[
      \epsilon_{m} = \frac{\sum_{n=1}^{N} w_{n}^{(m)} I(h_{m}(x) \neq t_{n})}{\sum_{n=1}^{N} w_{n}^{(m)}}
      \]
   c) Calculate a weighting coefficient for \( h_{m}(x) \):
      \[
      \alpha_{m} = ?
      \]
   d) Update the weighting coefficients:
      \[
      w_{n}^{(m+1)} = ?
      \]

How should we do this exactly?

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AdaBoost - Historical Development

• **Originally motivated by Statistical Learning Theory**
  - AdaBoost was introduced in 1996 by Freund & Schapire.
  - It was empirically observed that AdaBoost often tends not to overfit. (Breiman 96, Cortes & Drucker 97, etc.)
  - As a result, the margin theory (Schapire et al. 98) developed, which is based on loose generalization bounds.
    - Note: margin for boosting is *not* the same as margin for SVM.
    - A bit like retrofitting the theory...
  - However, those bounds are too loose to be of practical value.

• **Different explanation** (Friedman, Hastie, Tibshirani, 2000)
  - Interpretation as sequential minimization of an exponential error function (“Forward Stagewise Additive Modeling”).
  - Explains why boosting works well.
  - Improvements possible by altering the error function.
AdaBoost - Minimizing Exponential Error

- Exponential error function

\[ E = \sum_{n=1}^{N} \exp \{-t_n f_m(x_n)\} \]

- where \( f_m(x) \) is a classifier defined as a linear combination of base classifiers \( h_l(x) \):

\[ f_m(x) = \frac{1}{2} \sum_{l=1}^{m} \alpha_l h_l(x) \]

- Goal

  - Minimize \( E \) with respect to both the weighting coefficients \( \alpha_l \) and the parameters of the base classifiers \( h_l(x) \).
AdaBoost - Minimizing Exponential Error

• Sequential Minimization
  - Suppose that the base classifiers $h_1(x), \ldots, h_{m-1}(x)$ and their coefficients $\alpha_1, \ldots, \alpha_{m-1}$ are fixed.
  - Only minimize with respect to $\alpha_m$ and $h_m(x)$.

$$E = \sum_{n=1}^{N} \exp \left\{ -t_n f_m(x_n) \right\} \quad \text{with} \quad f_m(x) = \frac{1}{2} \sum_{l=1}^{m} \alpha_l h_l(x)$$

$$= \sum_{n=1}^{N} \exp \left\{ -t_n f_{m-1}(x_n) - \frac{1}{2} t_n \alpha_m h_m(x_n) \right\}$$

$$= \text{const.}$$

$$= \sum_{n=1}^{N} w_n^{(m)} \exp \left\{ -\frac{1}{2} t_n \alpha_m h_m(x_n) \right\}$$

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AdaBoost - Minimizing Exponential Error

\[ E = \sum_{n=1}^{N} w_n^{(m)} \exp \left\{ -\frac{1}{2} t_n \alpha_m h_m(x_n) \right\} \]

- **Observation:**
  - Correctly classified points: \( t_n h_m(x_n) = +1 \) \( \Rightarrow \) collect in \( T_m \)
  - Misclassified points: \( t_n h_m(x_n) = -1 \) \( \Rightarrow \) collect in \( F_m \)

- **Rewrite the error function as**

\[
E = e^{-\alpha_m/2} \sum_{n \in T_m} w_n^{(m)} + e^{\alpha_m/2} \sum_{n \in F_m} w_n^{(m)}
\]

\[
= \left( e^{\alpha_m/2} \right) \sum_{n=1}^{N} w_n^{(m)} I(h_m(x_n) \neq t_n)
\]

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AdaBoost - Minimizing Exponential Error

\[ E = \sum_{n=1}^{N} w^{(m)}_{n} \exp \left\{ -\frac{1}{2} t_{n} \alpha_{m} h_{m}(x_{n}) \right\} \]

- **Observation:**
  - Correctly classified points:  \( t_{n} h_{m}(x_{n}) = +1 \) \( \Rightarrow \) collect in \( \mathcal{T}_{m} \)
  - Misclassified points:  \( t_{n} h_{m}(x_{n}) = -1 \) \( \Rightarrow \) collect in \( \mathcal{F}_{m} \)

- **Rewrite the error function as**

\[ E = e^{-\alpha_{m}/2} \sum_{n \in \mathcal{T}_{m}} w^{(m)}_{n} + e^{\alpha_{m}/2} \sum_{n \in \mathcal{F}_{m}} w^{(m)}_{n} \]

\[ = \left( e^{\alpha_{m}/2} - e^{-\alpha_{m}/2} \right) \sum_{n=1}^{N} w^{(m)}_{n} I(h_{m}(x_{n}) \neq t_{n}) + e^{-\alpha_{m}/2} \sum_{n=1}^{N} w^{(m)}_{n} \]

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AdaBoost - Minimizing Exponential Error

• Minimize with respect to \( h_m(x) \):
  \[
  \frac{\partial E}{\partial h_m(x_n)} = 0
  \]
  \[
  E = \left( e^{\alpha_m/2} - e^{-\alpha_m/2} \right) \sum_{n=1}^{N} w_n^{(m)} I(h_m(x_n) \neq t_n) + e^{-\alpha_m/2} \sum_{n=1}^{N} w_n^{(m)}
  
  = \text{const.}
  \]

⇒ This is equivalent to minimizing

\[
J_m = \sum_{n=1}^{N} w_n^{(m)} I(h_m(x) \neq t_n)
\]

(our weighted error function from step 2a) of the algorithm)

⇒ We’re on the right track. Let’s continue...
AdaBoost - Minimizing Exponential Error

- Minimize with respect to $\alpha_m$:
  \[
  \frac{\partial E}{\partial \alpha_m} = 0
  \]

  \[
  E = \left( e^{\alpha_m/2} - e^{-\alpha_m/2} \right) \sum_{n=1}^{N} w^{(m)}_n I(h_m(x_n) \neq t_n) + e^{-\alpha_m/2} \sum_{n=1}^{N} w^{(m)}_n
  \]

  \[
  \left( \frac{1}{2} e^{\alpha_m/2} + \frac{1}{2} e^{-\alpha_m/2} \right) \sum_{n=1}^{N} w^{(m)}_n I(h_m(x_n) \neq t_n) = \frac{1}{2} e^{-\alpha_m/2} \sum_{n=1}^{N} w^{(m)}_n
  \]

  Weighted error $\epsilon_m := \frac{\sum_{n=1}^{N} w^{(m)}_n I(h_m(x_n) \neq t_n)}{\sum_{n=1}^{N} w^{(m)}_n} = \frac{e^{-\alpha_m/2}}{e^{\alpha_m/2} + e^{-\alpha_m/2}}$

  \[
  \epsilon_m = \frac{1}{e^{\alpha_m} + 1}
  \]

  ⇒ Update for the $\alpha$ coefficients:

  \[
  \alpha_m = \ln \left( \frac{1 - \epsilon_m}{\epsilon_m} \right)
  \]
AdaBoost - Minimizing Exponential Error

- Remaining step: update the weights
  - Recall that
    \[ E = \sum_{n=1}^{N} w_n^{(m)} \exp \left\{ -\frac{1}{2} t_n \alpha_m h_m(x_n) \right\} \]
    This becomes \( w_n^{(m+1)} \) in the next iteration.
  - Therefore
    \[ w_n^{(m+1)} = w_n^{(m)} \exp \left\{ -\frac{1}{2} t_n \alpha_m h_m(x_n) \right\} = \ldots = w_n^{(m)} \exp \{ \alpha_m I(h_m(x_n) \neq t_n) \} \]
    \( \Rightarrow \) Update for the weight coefficients.

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**AdaBoost - Final Algorithm**

1. **Initialization:** Set \( w_n^{(1)} = \frac{1}{N} \) for \( n = 1, \ldots, N \).

2. **For** \( m = 1, \ldots, M \) **iterations**
   
   a) Train a new weak classifier \( h_m(x) \) using the current weighting coefficients \( W^{(m)} \) by minimizing the weighted error function
   
   \[
   J_m = \sum_{n=1}^{N} w_n^{(m)} I(h_m(x) \neq t_n)
   \]
   
   b) Estimate the weighted error of this classifier on \( X \):
   
   \[
   \epsilon_m = \frac{\sum_{n=1}^{N} w_n^{(m)} I(h_m(x) \neq t_n)}{\sum_{n=1}^{N} w_n^{(m)}}
   \]
   
   c) Calculate a weighting coefficient for \( h_m(x) \):
   
   \[
   \alpha_m = \ln \left\{ \frac{1 - \epsilon_m}{\epsilon_m} \right\}
   \]
   
   d) Update the weighting coefficients:
   
   \[
   w_n^{(m+1)} = w_n^{(m)} \exp \{ \alpha_m I(h_m(x_n) \neq t_n) \}
   \]
AdaBoost - Analysis

• Result of this derivation
  ➢ We now know that AdaBoost minimizes an exponential error function in a sequential fashion.
  ➢ This allows us to analyze AdaBoost’s behavior in more detail.
  ➢ In particular, we can see how robust it is to outlier data points.
Comparing Error Functions

- Ideal misclassification error function (black)
  - This is what we want to approximate.
  - Unfortunately, it is not differentiable.
  \[ \Rightarrow \text{We cannot minimize it by gradient descent.} \]
Comparing Error Functions

- Ideal misclassification error function
- “Hinge error” used in SVMs
  - Zero error for points outside the margin ($z>1$).
  - Linearly increasing error for misclassified points ($z<1$).

Image source: Bishop, 2006
Comparing Error Functions

- Ideal misclassification error function
- “Hinge error” used in SVMs
- Exponential error function
  - Continuous approximation to ideal misclassification function.
  - Sequential minimization leads to simple AdaBoost scheme.
  - Disadvantage: exponential penalty for large negative values!

⇒ Less robust to outliers or misclassified data points!

B. Leibe

Image source: Bishop, 2006
Comparing Error Functions

- Ideal misclassification error function
- “Hinge error” used in SVMs
- Exponential error function
- “Cross-entropy error” \[ E = - \sum \{ t_n \ln y_n + (1 - t_n) \ln (1 - y_n) \} \]
  - Similar to exponential error for \( z > 0 \).
  - Only grows linearly with large negative values of \( z \).
  \( \Rightarrow \) Make AdaBoost more robust by switching \( \Rightarrow \) “GentleBoost”

Image source: Bishop, 2006
Summary: AdaBoost

• Properties
  - Simple combination of multiple classifiers.
  - Easy to implement.
  - Can be used with many different types of classifiers.
    - None of them needs to be too good on its own.
    - In fact, they only have to be slightly better than chance.
  - Commonly used in many areas.
  - Empirically good generalization capabilities.

• Limitations
  - Original AdaBoost sensitive to misclassified training data points.
    - Because of exponential error function.
    - Improvement by GentleBoost
  - Single-class classifier
    - Multiclass extensions available
Topics of This Lecture

- Ensembles of Classifiers
- Constructing Ensembles
  - Cross-validation
  - Bagging
- Combining Classifiers
  - Stacking
  - Bayesian model averaging
  - Boosting
- AdaBoost
  - Intuition
  - Algorithm
  - Analysis
  - Extensions
- Applications
Example Application: Face Detection

- Frontal faces are a good example of a class where global appearance models + a sliding window detection approach fit well:
  - Regular 2D structure
  - Center of face almost shaped like a “patch”/window

- Now we’ll take AdaBoost and see how the Viola-Jones face detector works
Feature extraction

“Rectangular” filters

Feature output is difference between adjacent regions

Efficiently computable with integral image: any sum can be computed in constant time

Avoid scaling images → scale features directly for same cost

Value at $(x,y)$ is sum of pixels above and to the left of $(x,y)$

Integral image

$$D = 1 + 4 - (2 + 3) = A + (A + B + C + D) - (A + C + A + B) = D$$

Slide credit: Kristen Grauman

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[Viola & Jones, CVPR 2001]
Large Library of Filters

Considering all possible filter parameters: position, scale, and type:

180,000+ possible features associated with each 24 x 24 window

Use AdaBoost both to select the informative features and to form the classifier

[Viola & Jones, CVPR 2001]
AdaBoost for Feature+Classifier Selection

- Want to select the single rectangle feature and threshold that best separates **positive** (faces) and **negative** (non-faces) training examples, in terms of **weighted** error.

**Resulting weak classifier:**

\[
h_t(x) = \begin{cases} 
+1 & \text{if } f_t(x) > \theta_t \\
-1 & \text{otherwise}
\end{cases}
\]

For next round, reweight the examples according to errors, choose another filter/threshold combo.

Slide credit: Kristen Grauman
AdaBoost for Efficient Feature Selection

- Image features = weak classifiers
- For each round of boosting:
  - Evaluate each rectangle filter on each example
  - Sort examples by filter values
  - Select best threshold for each filter (min error)
    - Sorted list can be quickly scanned for the optimal threshold
  - Select best filter/threshold combination
  - Weight on this features is a simple function of error rate
  - Reweight examples

Viola-Jones Face Detector: Results

Slide credit: Kristen Grauman
Viola-Jones Face Detector: Results

Slide credit: Kristen Grauman
Viola-Jones Face Detector: Results
References and Further Reading

- More information on Classifier Combination and Boosting can be found in Chapters 14.1-14.3 of Bishop’s book.

  Christopher M. Bishop
  Pattern Recognition and Machine Learning
  Springer, 2006

- A more in-depth discussion of the statistical interpretation of AdaBoost is available in the following paper: